Ch. 8

3. $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\phi^2$

a. $L\left(x^\alpha, \frac{dx^\alpha}{d\sigma}\right) \equiv \left[-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right]^{1/2}$

$$= \left[\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\sigma}\right)^2 - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{d\phi}{d\sigma}\right)^2\right]^{1/2}$$

b. From Lagrange's Eq. $-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{dx^\alpha}{d\sigma}\right)}\right) + \frac{\partial L}{\partial x^\alpha} = 0$

For $\alpha = 0$

$$-\frac{d}{d\sigma}\left[\frac{1}{L}\left(1-\frac{2M}{r}\right)\frac{dt}{d\sigma}\right] = 0 \qquad L = \frac{d\tau}{d\sigma}$$

$$-\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}\right] = 0$$

For $\alpha = 1$

$$-\frac{d}{d\sigma}\left[\frac{-1}{L}\left(1-\frac{2M}{r}\right)^{-1}\frac{dr}{d\sigma}\right] + \frac{1}{2}\frac{1}{L}\left[\frac{2M}{r^2}\left(\frac{dt}{d\sigma}\right)^2 + \frac{\frac{2M}{r^2}}{\left(1-\frac{2M}{r}\right)^2}\left(\frac{dr}{d\sigma}\right)^2 - 2r\left(\frac{d\phi}{d\sigma}\right)^2\right] = 0$$

$$-\frac{d}{d\tau}\left[-\left(1-\frac{2M}{r}\right)^{-1}\frac{dr}{d\tau}\right] + \frac{M}{r^2}\left(\frac{dt}{d\tau}\right)^2 + \frac{M/r^2}{\left(1-\frac{2M}{r}\right)^2}\left(\frac{dr}{d\tau}\right)^2 - r\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

For $\alpha = 2$

$$-\frac{d}{d\sigma}\left[\frac{1}{L}(-r^2)\frac{d\phi}{d\sigma}\right] = 0$$

$$-\frac{d}{d\tau}\left[-r^2\frac{d\phi}{d\tau}\right] = 0$$

11. $ds^2 = -dt^2 + dr^2 + r^2 d\phi^2$

Time independence $\Rightarrow$ Killing vector $\xi = (1,0,0)$

Azimuthal symmetry $\Rightarrow$ Killing vector $\eta = (0,0,1)$

$\therefore\ -\xi \cdot u \equiv e = \text{const.} \quad \eta \cdot u \equiv l = \text{const.}$

where $u^\alpha \equiv \frac{dx^\alpha}{d\lambda}$ for null geodesic