1. a. $t_{ff} = \left(\frac{3\pi}{32}\frac{1}{G\rho_0}\right)^{1/2}$ $\qquad \rho_0 = \mu m_H n = \frac{3M}{4\pi R^3}$

$$t_{cool} = \frac{3}{2}\frac{kT_{virial}}{n\Lambda} \qquad T_{virial} = \frac{\mu m_H \sigma^2}{3k} = \frac{GM\mu m_H}{5kR}$$

$$= \frac{3}{10}\frac{GM\mu m_H}{Rn\Lambda}$$

Equating

$$t_{ff} = t_{cool}$$

$$\left(\frac{3\pi}{32}\frac{1}{G\rho_0}\right)^{1/2} = \frac{3}{10}\frac{GM\mu m_H}{Rn\Lambda}$$

$$\frac{\cancel{3}\pi}{8\,\cancel{32}}\frac{\cancel{4}\pi R^3}{\cancel{3}GM} = \frac{9}{100}\frac{G^2M^2\mu^4 m_H^4}{R^2\rho_0^2\Lambda^2}$$

$$\frac{\cancel{\pi^2}}{8}\frac{\cancel{R^3}}{6M} = \frac{\cancel{9}}{100}\frac{G^2\cancel{M^2}\mu^4 m_H^4}{\cancel{R^2}\Lambda^2}\frac{16\cancel{\pi^2}R^{\cancel{6}}}{\cancel{9M^2}}$$

$$M = \frac{25}{32}\frac{\Lambda^2}{G^3\mu^4 m_H^4 R}$$

b. $\Lambda = 10^{-37}\ \mathrm{W\,m^3}$ $\qquad R = 60\,\mathrm{kpc} = 6\times10^4\,\mathrm{pc} = 1.85\times10^{21}\,\mathrm{m}$

$$M = \frac{25}{32}\frac{(10^{-37}\ \mathrm{W\,m^3})^2}{(6.67\times10^{-11}\ \frac{\mathrm{N\,m^2}}{\mathrm{kg^2}})^3(0.6)^4(1.67\times10^{-27}\,\mathrm{kg})^4(1.85\times10^{21}\,\mathrm{m})}$$

$$= 1.41\times10^{43}\,\mathrm{kg} = 7.09\times10^{12}\,M_\odot$$

2. $M_{LMC} = 2\times10^{10}\,M_\odot$ $\qquad R = 51\,\mathrm{kpc} = 5.1\times10^4\,\mathrm{pc} = 1.58\times10^{21}\,\mathrm{m}$

a. $M_{MW} \approx 5.4\times10^{11}\,M_\odot$

$$r = R\left(\frac{M_{LMC}}{2M_{MW}}\right)^{1/3} = 51\,\mathrm{kpc}\left(\frac{2\times10^{10}M_\odot}{2(5.4\times10^{11}M_\odot)}\right)^{1/3}$$

$$= 13.5\ \mathrm{kpc}$$

b. Size of LMC

$$r = 51\,\text{kpc} \cdot 460' \,\frac{1^\circ}{60'} \cdot \frac{2\pi\,\text{rad}}{360^\circ} = 6.8\,\text{kpc}$$

Tidal radius is about twice as large as actual size.

3. $$\xi(M) = \frac{dN}{dM} = CM^{-(1+x)}$$

a. $x = 1.8$

$$N_{2\text{-}3} = \int dN = \int_{2M_\odot}^{3M_\odot} CM^{-2.8}\,dM = \frac{-C}{1.8} M^{-1.8}\Big|_{2M_\odot}^{3M_\odot}$$

$$N_{10\text{-}11} = -\frac{C}{1.8} M^{-1.8}\Big|_{10M_\odot}^{11M_\odot}$$

$$\frac{N_{2\text{-}3}}{N_{10\text{-}11}} = \frac{3^{-1.8} - 2^{-1.8}}{11^{-1.8} - 10^{-1.8}} = 59.5$$

b. $$\frac{dN}{dL} = \frac{dN}{dM}\frac{dM}{dL} = CM^{-(1+x)} \frac{d}{dL}\left(M_\odot\left(\frac{L}{L_\odot}\right)^{1/\alpha}\right)$$

$$= C' L^{\frac{-(1+x)}{\alpha}} L^{\frac{1}{\alpha}-1} = C' L^{-\frac{(x+\alpha)}{\alpha}}$$

c. $x = 1.8, \alpha = 4$

$$N_{2\text{-}3} = \int_{2L_\odot}^{3L_\odot} C' L^{-1.45}\,dL = \frac{-C'}{0.45} L^{-0.45}\Big|_{2L_\odot}^{3L_\odot}$$

$$N_{10\text{-}11} = \frac{-C'}{0.45} L^{-0.45}\Big|_{10L_\odot}^{11L_\odot}$$

$$\frac{N_{2\text{-}3}}{N_{10\text{-}11}} = \frac{3^{-0.45} - 2^{-0.45}}{11^{-0.45} - 10^{-0.45}} = 8.20$$

d. Luminosity rises faster than mass does along the main sequence, so a star only a few times more massive can be many times more luminous.
Also illustrates that low mass/low luminosity stars are more common than high mass/high luminosity.